B. Sc. Part II (Hons) 4th Paper

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Vector calculus (continued)

Vector triple product

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

This is so called because it is a vector quantity, obtained from product of 3 vectors.

$$(\vec{b}\times\vec{c})\times\vec{a} = (\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b}$$

Impt →

$$\boxed{\begin{array}{l} \vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c} \\ \text{Also, } (\vec{a}\cdot\vec{c})\vec{b} \neq \vec{a}(\vec{c}\cdot\vec{b}) \neq (\vec{a}\cdot\vec{b})\vec{c} \end{array}}$$

Q Prove That $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

Solution

$$\because \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}$$

$$\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$$

Adding the above 3 equations, and using the ~~commutative~~ property $(\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$$\therefore \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$$

Scalar product of 4 vectors

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

Proof

Let $\vec{c}\times\vec{d} = \vec{p}$

$\therefore$ LHS $= (\vec{a}\times\vec{b})\cdot\vec{p}$

$= \vec{a}\cdot(\vec{b}\times\vec{p})$ [inter-changeability of dot & cross]

$= \vec{a}\cdot\{\vec{b}\times(\vec{c}\times\vec{d})\}$

$= \vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}]$

$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d} \end{vmatrix} = \text{RHS}$$

Vector product of four vectors

Ex: $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) \rightarrow$ It is a vector quantity

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Proof Let $\vec{c}\times\vec{d} = \vec{m}$

$$LHS = (\vec{a}\times\vec{b})\times\vec{m} = (\vec{a}\cdot\vec{m})\vec{b} - (\vec{b}\cdot\vec{m})\vec{a}$$

$$= [\vec{a}\cdot(\vec{c}\times\vec{d})]\vec{b} - \{\vec{b}\cdot(\vec{c}\times\vec{d})\}\vec{a}$$

$$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

[1st part proved]

Again let $\vec{a}\times\vec{b} = \vec{m}$ then

$$LHS = \vec{m}\times(\vec{c}\times\vec{d}) = (\vec{m}\cdot\vec{d})\vec{c} - (\vec{m}\cdot\vec{c})\vec{d}$$

$$= \{(\vec{a}\times\vec{b})\cdot\vec{d}\}\vec{c} - \{(\vec{a}\times\vec{b})\cdot\vec{c}\}\vec{d}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

2nd part proved

Q. Prove that

$$\left[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

$$\text{LHS} = (\vec{a}+\vec{b})\cdot\left\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\right\}$$

$$= (\vec{a}+\vec{b})\cdot\left\{\vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{c}+\vec{a})\right\}$$

$$= (\vec{a}+\vec{b})\cdot\left\{\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}\right\}$$

$$= (\vec{a}+\vec{b})\cdot\left\{\vec{b}\times\vec{c} - \vec{a}\times\vec{b} + \vec{c}\times\vec{a}\right\}$$

$$= \vec{a}\cdot\left\{\vec{b}\times\vec{c} - \vec{a}\times\vec{b} + \vec{c}\times\vec{a}\right\}$$
$$+ \vec{b}\cdot\left\{\vec{b}\times\vec{c} - \vec{a}\times\vec{b} + \vec{c}\times\vec{a}\right\}$$

$$= \vec{a}\cdot(\vec{b}\times\vec{c}) - \vec{a}\cdot(\vec{a}\times\vec{b}) + \vec{a}\cdot(\vec{c}\times\vec{a})$$
$$+ \vec{b}\cdot(\vec{b}\times\vec{c}) - \vec{b}\cdot(\vec{a}\times\vec{b}) + \vec{b}\cdot(\vec{c}\times\vec{a})$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] - \left[\vec{a}\ \vec{a}\ \vec{b}\right] + \left[\vec{a}\ \vec{c}\ \vec{a}\right]$$
$$+ \left[\vec{b}\ \vec{b}\ \vec{c}\right] - \left[\vec{b}\ \vec{a}\ \vec{b}\right] + \left[\vec{b}\ \vec{c}\ \vec{a}\right]$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] - 0 + 0 + 0 - 0 + \left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

$$= 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

$$= \text{RHS}$$

Q: Prove that

$$\left[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\right] = \left[\vec{a}\ \vec{b}\ \vec{c}\right]^2$$

Soln

$$LHS = (\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$$

$$= (\vec{a}\times\vec{b})\cdot\left\{\left[\vec{b}\ \vec{c}\ \vec{a}\right]\vec{c} - \left[\vec{c}\ \vec{c}\ \vec{a}\right]\vec{b}\right\}$$

(by definition of vector product of 4 vectors)

$$= (\vec{a}\times\vec{b})\cdot\left\{\left[\vec{b}\ \vec{c}\ \vec{a}\right]\vec{c} - 0.\vec{b}\right\}$$

$$\left[\because \left[\vec{c}\ \vec{c}\ \vec{a}\right] = 0\right]$$

$$= (\vec{a}\times\vec{b})\cdot\left[\vec{b}\ \vec{c}\ \vec{a}\right]\vec{c}$$

$$= \{(\vec{a}\times\vec{b})\cdot\vec{c}\}\left[\vec{b}\ \vec{c}\ \vec{a}\right]$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]^2$$ Proved

Q Prove that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$ iff $\vec{a}$ and $\vec{c}$ are collinear.

Solution : $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

Scanned with CamScanner

Necessary part Given that

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$$

$$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$$

$$\Rightarrow (\vec{a}\cdot\vec{b})\vec{c} = (\vec{b}\cdot\vec{c})\vec{a}$$

$$\Rightarrow (\vec{a}\cdot\vec{b})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} = \vec{0}$$

$$\Rightarrow (\vec{a}\cdot\vec{b})\vec{c} - (\vec{c}\cdot\vec{b})\vec{a} = \vec{0}$$

$$\Rightarrow (\vec{a}\times\vec{c})\times\vec{b} = \vec{0}$$

$\Rightarrow$ Either $\vec{a}\times\vec{c} = \vec{0}$ or $\vec{b} = \vec{0}$

but $\vec{b} \neq \vec{0} \Rightarrow \vec{a}\times\vec{c} = \vec{0}$

$\Rightarrow \vec{a}$ and $\vec{c}$ are collinear.

Sufficiency Given that $\vec{a}$ and $\vec{c}$ are collinear. ~~Let so~~ Then

$$\vec{c} = k\vec{a} \Rightarrow \vec{a}\times\vec{c} = \vec{a}\times k\vec{a} = k\,\vec{a}\times\vec{a} = \vec{0}$$

we have to prove that

$$(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$$

$$\text{LHS} = (\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$$

$$= (\vec{a}\cdot k\vec{a})\vec{b} - (\vec{b}\cdot k\vec{a})\vec{a}$$

$$= k(\vec{a}\cdot\vec{a})\vec{b} - k(\vec{a}\cdot\vec{b})\vec{a}$$

$$\text{RHS} = \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$= (\vec{a}\cdot k\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})k\vec{a}$$

$$= k(\vec{a}\cdot\vec{a})\vec{b} - k(\vec{a}\cdot\vec{b})\vec{a}$$

$\therefore$ LHS = RHS

sufficent part proved